Midterm Exam

MAS501 Analysis for Engineers, Spring 2011

♠ 공지사항 ♠

- 휴대전화는 진동으로 맞춰 주시고 시간을 확인하는 용도로만 사용해 주십시오.
- 학번과 이름을 기입하신 후 답안지는 양면으로 작성해 주십시오.
- 강의 및 숙제를 통해 피드백 해드린 내용을 잘 이해하셨으리라 믿고 중간고사 채점은 숙제 채점보다 엄격히 할 예정입니다. 우아하고 정확한 답안을 기대하겠습니다.

0. (5pts) 기말고사 날짜 및 시간 선정에 관한 설문조사입니다. 5월 23일–27일 학기말 시험 기간 동안 중 *월요일, 수요일, 금요일 오전 9시나 오전 10시*에 두 시간 동안 기말고사를 보려 합니다. 불가능한 요일과 시간을 적고 엑스 표시를, 가능하지만 원하지 않는 요일과 시간을 적고 세모 표시를 해주세요.

1. (20pts) Determine whether the following statements are true or false. (You don't need to prove or disprove.) If you are correct, you gain two points. If you are wrong, you *lose* two points. If you don't write the answer, nothing happens.

   (a) If you are the Batman, I'm the Superman. (I'm assuming you're not the Batman.)

   (b) The set of all polynomials with rational coefficients is countable.

   (c) Suppose E is a subset of the metric space Ω. Then $x \in \bar{E}$ if and only if there is a sequence of points $x_n \in E$ with $x_n \to x$.

   (d) A countable intersection of open sets is open.

   (e) Let A and B be nonempty sets of real numbers. Then

   $$\inf(A+B) = \inf A + \inf B$$

   where $A + B := \{a + b : a \in A,\, b \in B\}$.

   (f) Every Cauchy sequence is bounded.

   (g) Let $\{a_n\}$ be a sequence of real numbers. If $\sum a_n$ converges absolutely, then

   $$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1.$$

   (h) If $f : \Omega \to \Omega'$ and $\{A_i\}$ is an arbitrary family of subsets of Ω, then

   $$f\Big(\bigcap_i A_i\Big) = \bigcap_i f(A_i).$$

   (i) If $f : \mathbf{R} \to \mathbf{R}$ is differentiable everywhere, then f' cannot have a simple discontinuity.

   (j) Suppose that $f : \mathbf{R} \to \mathbf{R}$ is differentiable everywhere and

   $$\lim_{x\to\infty} \big(f(x) + f'(x)\big) = 1.$$

   Then $\lim_{x\to\infty} f(x) = 1$.

2. Let $\{x_n\}$ be a sequence in the metric space Ω. Prove the following statements:

   (a) (10pts) If $x_n \to a$ and $x_n \to b$ then $a = b$.

   (b) (10pts) If $\{x_n\}$ is Cauchy with a subsequence converging to x, then $x_n \to x$.

3. (15pts) Let f be a continuous mapping from Ω to Ω', where Ω and Ω' are metric spaces. Show that if Ω is compact and $\{x_n\}$ is a Cauchy sequence in Ω, then $\{f(x_n)\}$ is a Cauchy sequence in Ω'.

4. A very special agent DiNozzo proved the following *generalized* Bolzano-Weierstrass theorem:

   "Let $x_1, x_2, \cdots$ be a bounded sequence in the metric space Ω. Then there is a subsequence $x_{n_1}, x_{n_2}, \cdots$ converging to a point x in Ω."

   DiNozzo's proof. We may assume that all the x_n belong to a fixed closed ball $C_r(a)$. By the Heine-Borel theorem, $C_r(a)$ is compact, and the result follows from the compactness of $C_r(a)$. $\square$

   (a) (5pts) What is wrong with his argument?

   (b) (10pts) Give a metric space in which the *generalized* Bolzano-Weierstrass theorem is wrong. You don't need to prove your space is a *metric* space. But you should prove that the theorem is not true in your space.

5. (10pts) Let $\{a_n\}$ be a sequence of positive real numbers. Prove that

$$\limsup_{n\to\infty} \left(\frac{a_{n+1}+1}{a_n}\right)^n \geq 1.$$

6. (15pts) Suppose

   (a) $f : [0, \infty) \to \mathbf{R}$ is continuous.

   (b) $f'(x)$ exists on $(0, \infty)$.

   (c) $f(0) = 0$.

   (d) $f'(x)$ is increasing on $(0, \infty)$.

   Prove that $f(x)/x$ is increasing on $(0, \infty)$.

Bonus! (5pts) Find an increasing function in $\mathbf{R}$ whose set of discontinuities is precisely $\mathbf{Q}$. (DO NOT justify your answer.)